Available online at www.sciencedirect.com
SCIENCE DIRECT
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 48 (2004) 1477–1484
www.elsevier.com/locate/camwa

Convergence and Periodicity of Solutions for a Class of Delay Difference Equations

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(Received August 2003; accepted July 2004)

Abstract—We derive a class of delay difference equations with piecewise constant nonlinearity. The convergence of solutions and the existence of asymptotically stable periodic solutions are investigated, for such a class of difference equations.

Keywords—Difference equation, Convergence, Periodicity.

1. INTRODUCTION

The qualitative behavior of solutions of differential equations and delay difference equations with piecewise constant nonlinearity has been received many investigations. The reader may consult the references listed in the paper and those contained therein. As mentioned in [1–3], the main attraction of studying such equations is the fact that they represent a hybrid of continuous and discrete dynamical systems and exhibit the properties of both differential and difference equations.

In this paper, we consider the following delay difference equation,

$$x_n = ax_{n-1} + (1-a) f(x_{n-k}), \qquad n = 0, 1, 2, \dots, \tag{1.1}$$

where $a \in (0,1), k$ is a positive integer and $f : R \to R$ is the function given by the piecewise constant nonlinearity,

$$f(\xi) = \begin{cases} 1, & \xi \in (b, c], \\ 0, & \xi \in (-\infty, b] \cup (c, \infty), \end{cases} \tag{1.2}$$

for constants $b, c \in [0, \infty)$ and $b < c$.

Research supported by the Natural Science Foundation of China (10371034), the Key Project of Chinese Ministry of Education (No.[2002] 78), the Doctor Program Foundation of Chinese Ministry of Education (20010532002). We wish to thank two reviewers for their valuable comments that led to truly significant improvement of the manuscript.

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0898-1221/04/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2004.07.005

Equation (1.1) can be regarded as a normal form of the following equation,

$$x_n = ax_{n-1} + \lambda f(x_{n-k}), \qquad n = 0, 1, 2, \dots, \tag{1.3}$$

where $a \in (0,1)$ and $\lambda > 0$ are constants, $f : R \to R$ is the function given by (1.2).

In fact, we may rescale the variables and parameters in (1.2) and (1.3) by

$$x_n^* = \frac{1-a}{\lambda} x_n, \qquad f^*(\xi) = f\left(\frac{\lambda}{1-a}\xi\right), \qquad b^* = \frac{1-a}{\lambda} b, \qquad c^* = \frac{1-a}{\lambda} c$$

for $n = 0, 1, 2, \dots$, and then, drop the $*$ to get (1.1) with f satisfying (1.2).

On the other hand, equation (1.3) can be derived from the following delay differential equation with a piecewise constant argument,

$$\frac{dx}{dt} = -\mu x + \beta f(x([t-l])), \qquad t \geq 0, \tag{1.4}$$

where $\mu > 0$ and $\beta > 0$ are given constants, l is a nonnegative integer, $f : R \to R$ is the function given by (1.2), and $[\cdot]$ denotes the greatest integer function.

As we know, equation (1.4) has also wide applications in certain biomedical models. For some background on (1.4) and on some other systems of differential equations with piecewise constant arguments, we refer to [4,5]. It is not difficult to discretize (1.4) into a difference equation (1.3). In fact, we may rewrite (1.4) in the following form,

$$\frac{d}{dt}\left(x(t) e^{\mu t}\right) = e^{\mu t} \beta f(x([t-l])), \qquad t \geq 0. \tag{1.5}$$

Let n be a nonnegative integer and $k = l + 1$. Then, we integrate (1.5) from $n-1$ to t, with $t \in [n-1, n)$, to obtain

$$x(t) e^{\mu t} - x(n-1) e^{\mu(n-1)} = \frac{\beta}{\mu}\left(e^{\mu t} - e^{\mu(n-1)}\right) f(x(n-k)). \tag{1.6}$$

Let $t \to n$. Then, it follows from (1.5) that

$$x(n) = e^{-\mu} x(n-1) + \frac{\beta}{\mu}\left(1 - e^{-\mu}\right) f(x(n-k)), \qquad n = 0, 1, 2, \dots.$$

Let $x_n = x(n)$. Then, we obtain

$$x_n = e^{-\mu} x_{n-1} + \frac{\beta}{\mu}\left(1 - e^{-\mu}\right) f(x_{n-k}), \qquad n = 0, 1, 2, \dots,$$

which is a special case of (1.3) with $a = e^{-\mu}$ and $\lambda = (\beta/\mu)(1 - e^{-\mu})$.

Our goal is to discuss the convergence and periodic properties of solutions of delay difference equation $\{(1.1), (1.2)\}$.

For the sake of simplicity, let N denote the set of all nonnegative integers. For any $m, n \in N$, define $N(m) = \{m, m+1, \dots\}$ and $N(m,n) = \{m, m+1, \dots, n\}$, $(m \leq n)$. Obviously, $N = N(0)$. A solution of (1.1) is a sequence $\{x_n\}$ in R with $n \in N(-k)$ that satisfy the equation, $\{(1.1), (1.2)\}$, for $n \in N$. Let X denote the set of mappings from $N(-k,-1)$ to R. Clearly, for any $\varphi \in X$, equation (1.1) has a unique solution $\{x_n\}_{n=-k}^{+\infty}$ satisfying the initial conditions,

$$x_i = \varphi(i), \qquad \text{for } i \in N(-k,-1). \tag{1.7}$$

Throughout the remaining part, we denote the solution of (1.1) with the initial condition (1.7) by $\{x_n\}_{n=-k}^{+\infty}$. We shall concentrate on the case that $\varphi - b$ and $\varphi - c$ have no sign changes on $N(-k,-1)$. More precisely, we consider the case, $\varphi \in X_1 \cup X_2 \cup X_3 = X_{b,c} \subset X$, where

$$X_1 = \{\varphi | \varphi : N(-k,-1) \to R \text{ and } \varphi(i) \leq b, \text{ for } i \in N(-k,-1)\},$$
$$X_2 = \{\varphi | \varphi : N(-k,-1) \to R \text{ and } b < \varphi(i) \leq c, \text{ for } i \in N(-k,-1)\},$$

and

$$X_3 = \{\varphi | \varphi : N(-k,-1) \to R \text{ and } \varphi(i) > c, \text{ for } i \in N(-k,-1)\}.$$

2. CONVERGENCE

In this section, we consider the convergence of solutions for (1.1).

THEOREM 2.1. *The following conclusions are true.*

(i) *If $\varphi \in X_1$ and $b \geq 1$, then, $x_n \to 0$, as $n \to \infty$.*
(ii) *If $\varphi \in X_2$ and $b > 1$, then, $x_n \to 0$, as $n \to \infty$,*
(iii) *If $\varphi \in X_2$ and $b = 1$, then, $x_n \to 1$, as $n \to \infty$,*
(iv) *If $\varphi \in X_3$, $b > 1$, and $c \geq ba^{-k}$, then, $x_n \to 0$, as $n \to \infty$,*
(v) *If $\varphi \in X_3$, $b = 1$, and $c \geq a^{-k}$, then, $x_n \to 1$, as $n \to \infty$.*

PROOF. First, we will show Conclusion (i). Since $\varphi \in X_1$, from (1.1) and (1.2), it follows that

$$x_n - ax_{n-1} = 0, \tag{2.1}$$

for $n \in N(0, k-1)$. Therefore,

$$x_n = \varphi(-1)a^{n+1}, \tag{2.2}$$

for $n \in N(0, k-1)$. If $0 < \varphi(-1) \leq b$, then, $x_n = \varphi(-1)a^{n+1} < \varphi(-1) \leq b$, for $n \in N(0, k-1)$, and if $\varphi(-1) \leq 0$, then, $x_n = \varphi(-1)a^{n+1} \leq 0 \leq b$, for $n \in N(0, k-1)$. This implies that $x_n \in X_1$ for $n \in N(0, k-1)$, and that (2.1) is satisfied, for $n \in N(k, 2k-1)$. Hence, we have $x_n \in X_1$ for $n \in N(k, 2k-1)$. Repeating this procedure, we can obtain that x_n satisfies (2.2), for all $n \in N$. Consequently, we derive that $x_n \to 0$ as $n \to \infty$.

Second, we shall consider Conclusion (ii). Let $\varphi \in X_2$ and $b > 1$. By using (1.1) and (1.2), we get

$$x_n = (\varphi(-1) - 1)\, a^{n+1} + 1, \qquad \text{for } n \in N(0, k-1). \tag{2.3}$$

Let n_1 be the least nonnegative integer, such that

$$x_{n_1 - 1} > b \qquad \text{and} \qquad x_{n_1} \leq b. \tag{2.4}$$

Then, (2.3) holds, for $n \in N(0, n_1 + k - 1)$.

By (2.3) and (2.4), we have

$$x_{n_1+i} = (\varphi(-1) - 1)\, a^{n_1+i+1} + 1 < (\varphi(-1) - 1)\, a^{n_1+1} + 1 = x_{n_1} \leq b,$$

for $i \in N(0, k-1)$, which implies $x_{n_1+i} \in X_1$, for $i \in N(0, k-1)$. By Conclusion (i), we have $x_n \to 0$, as $n \to \infty$.

We will next prove Conclusion (iii). Let $\varphi \in X_2$ and $b = 1$. In view of (1.1) and (1.2), we see that

$$x_n = ax_{n-1} + 1 - a, \tag{2.5}$$

for $n \in N(0, k-1)$. It follows that

$$x_n = (\varphi(-1) - 1)\, a^{n+1} + 1, \tag{2.6}$$

for $n \in N(0, k-1)$.

Noting

$$b = 1 < x_n = (\varphi(-1) - 1)\, a^{n+1} + 1 \leq (c-1)\ a^{n+1} + 1 < c,$$

for $n \in N(0, k-1)$, we establish (2.5), for $n \in N(k, 2k-1)$. Therefore, we have $b < x_n < c$, for $n \in N(k, 2k-1)$. Repeating this procedure, we can obtain that x_n satisfies (2.5), for all $n \in N$, and then, $x_n \to 1$, as $n \to \infty$.

We shall finally discuss Conclusions (iv) and (v). Let $\varphi \in X_3$ and $c \geq ba^{-k}$. Since $\varphi \in X_3$, from (1.1) and (1.2), it follows that

$$x_n = \varphi(-1)\, a^{n+1}, \tag{2.7}$$

for $n \in N(0, k-1)$. Let m_1 be the least nonnegative integer, such that

$$x_{m_1 - 1} > c \qquad \text{and} \qquad x_{m_1} \leq c. \tag{2.8}$$

Then, (2.7) holds, for $n \in N(0, m_1 + k - 1)$.

Combining (2.7) with (2.8) leads to

$$x_{m_1+i} = \varphi(-1)\, a^{m_1+i+1} < \varphi(-1)\, a^{m_1+1} = x_{m_1} \leq c, \qquad \text{for } i \in N(0, k-1).$$

Noting $c \geq ba^{-k}$ and (2.8), we easily verify that

$$x_{m_1+i} = \varphi(-1)\, a^{m_1+i+1} \geq \varphi(-1)\, a^{m_1+k} = x_{m_1-1} a^k > ca^k \geq b, \qquad \text{for } i \in N(0, k-1).$$

Therefore, we have $x_{m_1+i} \in X_2$ for $i \in N(0, k-1)$. By Conclusions (ii) and (iii), we can obtain Conclusions (iv) and (v). This completes the proof of Theorem 2.1.

THEOREM 2.2. *Let $0 \leq b < 1 \leq c$. Then,*

(i) *$x_n \to 0$, as $n \to \infty$ for $\varphi \in X_1$,*
(ii) *$x_n \to 1$, as $n \to \infty$, for $\varphi \in X_2$ or $\varphi \in X_3$ and $c > ba^{-k}$.*

PROOF. The proof of Conclusion (i) is very similar to that of Conclusion (i) of Theorem 2.1, and henceforth, is left as exercises.

We shall prove Conclusion (ii). To this end, we let $\varphi \in X_2$ or $\varphi \in X_3$ and $c > ba^{-k}$ and discuss it in the following cases.

CASE 1. Let $\varphi \in X_2$. By using (1.1) and (1.2), we show that

$$x_n = (\varphi(-1) - 1)\, a^{n+1} + 1, \tag{2.9}$$

for $n \in N(0, k-1)$. If $b < \varphi(-1) \leq 1$, then,

$$b < (b-1)\, a^{n+1} + 1 < x_n = (\varphi(-1) - 1)\, a^{n+1} + 1 \leq 1 < c, \qquad \text{for } n \in N(0, k-1),$$

and, if $1 < \varphi(-1) \leq c$, then,

$$b < 1 < x_n = (\varphi(-1) - 1)\, a^{n+1} + 1 \leq (c-1)\, a^{n+1} + 1 < c, \qquad \text{for } n \in N(0, k-1).$$

This implies that $x_n \in X_2$, for $n \in N(0, k-1)$, and that (2.9) is satisfied, for $n \in N(k, 2k-1)$. Repeating this procedure, we can obtain that x_n satisfies (2.9), for all $n \in N$, and then, $x_n \to 1$, as $n \to \infty$.

CASE 2. Let $\varphi \in X_3$ and $c \geq ba^{-k}$. We can verify Case 2 by combining the proof method which is similar to that of Conclusions (iv) and (v) of Theorem 2.1 with Case 1, and henceforth, is left as exercises. We finish the proof of Theorem 2.2.

THEOREM 2.3. *Assume that $0 \leq b < c < 1$, then, $x_n \to 0$, as $n \to \infty$, for $\varphi \in X_1$.*

PROOF. The proof of Theorem 2.3 is very similar to that of Conclusion (i) of Theorem 2.1, and hence, is omitted.

3. EXISTENCE OF PERIODIC SOLUTIONS

The following lemma is helpful for discussing existence and attraction of periodic solutions.

LEMMA 3.1. *Let $0 \le b < ba^{-k} < c < 1$. For any solution $\{x_n\}$ of (1.1) with initial value $\varphi \in X_2 \cup X_3$, there exists integers m_1 and $m \in N(-1)$ with $m - m_1 \ge k$, such that $x_n \in X_2$, for $n \in N(m_1, m)$ and $x_{m+1} \in (c, 1)$.*

PROOF. We shall verify this lemma by several cases.

CASE 1. Let $\varphi \in X_2$. We shall show that there exists an $n_0 \in N(-1)$, such that $x_n \in X_2$, for $n \in N(-k, n_0)$ and $x_{n_0+1} \notin X_2$. Otherwise, we have $x_n \in X_2$, for any $n \in N(-k)$. It follows from (1.1) and (1.2) that

$$x_n = (\varphi(-1) - 1)\, a^{n+1} + 1, \qquad \text{for } n \in N, \tag{3.1}$$

which implies that $x_n \to 1 > c$, as $n \to \infty$. It contradicts the fact that $x_n \in X_2$, for $n \in N(-k)$.

Note that

$$x_{n_0+1} = ax_{n_0} + (1-a) > b,$$

which implies that $x_{n_0+1} \in X_3$.

Moreover,

$$x_{n_0+1} = ax_{n_0} + (1-a) \le ac + (1-a) < 1,$$

so, the conclusion of Lemma 3.1 holds, where $m = n_0$ and $m_1 = -k$.

CASE 2. Let $\varphi \in X_3$. For this case, by the similar argument as that of the proof of Conclusions (iv) and (v) of Theorem 2.1, it is easy to verify that there exists some $n_1 \in N(-1)$, such that $x_n \in X_2$, for $n \in N(n_1, n_1 + k - 1)$. Hence, Case 2 can be proved by using Case 1.

REMARK 3.1. *From the proof of Lemma 3.1, we see that to study the limiting behavior of solutions with initial values in $X_2 \cup X_3$, for $c \in (ba^{-k}, 1)$, it suffices to restrict initial function $\varphi \in X_2$ and the first iteration, $x_0 \in D_0 = (c, 1)$.*

THEOREM 3.1. *For $p, q \in N$, define*

$$\begin{aligned}
\Delta_1 &= \max\left\{a^{p+1}, 1 + \left(b - a^{p+1}\right) a^{-(p+k)}\right\},\\
\Delta_2 &= \max\left\{1 - a^q + a^{p+q+k}, \left[b - 1 + a^q\left(1 - a^{k+p} + a^{2k+p-1}\right)\right]\left(a^{q+2k+p-1}\right)^{-1}\right\},\\
I_1(p,q) &= \left(\Delta_1, \frac{a^p\left(1 - a^{k-1}\right)}{1 - a^{k+p-1}}\right),\\
I_2(p,q) &= \left(\Delta_2, 1 - \frac{a^{q+1}\left(1 - a^{k+p}\right)}{1 - a^{2k+p+q}}\right).
\end{aligned}$$

Let $0 \le b < ba^{-k} < c < 1$ and $c \in I_1(p,q) \cap I_2(p,q)$, for some $p, q \in N$. Then, equation (1.1) exists an asymptotically stable periodic solution $\{x'_n\}$ with initial function $\varphi \in X_2 \cup X_3$, whose minimal period is $2k + p + q$.

PROOF. From Remark 3.1, it suffices to consider $\varphi \in X_2$ and the first iteration,

$$x_0 \in D_0 = (c, 1).$$

Note that the iteration of the linear map,

$$g_1(u) = au + 1 - a, \tag{3.2}$$

satisfies

$$g_1^{(n)}(u) = a^n u + 1 - a^n, \tag{3.3}$$

and that the iteration of the linear map,

$$g_2(u) = au, \tag{3.4}$$

satisfies

$$g_2^{(n)}(u) = a^n u. \tag{3.5}$$

Let $g_1^{(n)}(D_0) = D_n$, for $n \in N(1, k-1)$. Since $\varphi \in X_2$ and $x_0 \in D_0 = (c, 1)$, it is clear that the solution $\{x_n\}$ of $\{(1.1), (1.2)\}$ satisfies

$$x_n = g_1^{(n)}(x_0), \qquad \text{for } n \in N(1, k-1). \tag{3.6}$$

Moreover, it is easy to prove that

$$D_n = \left(g_1^{(n)}(c), g_1^{(n)}(1)\right), \qquad \text{for } n \in N(1, k-1). \tag{3.7}$$

In view of $u \in (0,1)$, we have

$$1 > g_1^{(k)}(u) > g_1^{(k-1)}(u) > \cdots > g_1^{(0)}(u) = u. \tag{3.8}$$

Recalling that $c \in (ba^{-k}, 1) \subset (0, 1)$, and combining (3.7) with (3.8), we obtain that $D_n \subset X_3$, for all $n \in N(0, k-1)$.

Let n_1 be the largest integer, such that $x_n \in X_3$, for $n \in N(0, n_1 + k - 1)$. Then, from (1.1) and (1.2), we can obtain

$$x_{n+k-1} = g_2^{(n)}(x_{k-1}) = g_2^{(n)} \cdot g_1^{(k-1)}(x_0), \qquad \text{for } n \in N(1, n_1 + k), \tag{3.9}$$

which implies that $x_{n+k-1} \in D_{n+k-1}$, for $n \in N(1, n_1 + k)$, where

$$D_{n+k-1} = g_2^{(n)} \cdot g_1^{(k-1)}(D_0), \qquad \text{for } n \in N(1, n_1 + k).$$

Furthermore, it follows from (3.7) that

$$D_{n+k-1} = \left(g_2^{(n)} \cdot g_1^{(k-1)}(c), g_2^{(n)} \cdot g_1^{(k-1)}(1)\right) = \left(a^{n+k-1}c - a^{n+k-1} + a^n, a^n\right), \tag{3.10}$$

for $n \in N(1, n_1 + k)$. Since $c \in I_1(p, q)$, we can verify that from (3.10),

$$D_{n+k-1} \subset X_3, \qquad \text{for } n \in (0, p), \tag{3.11}$$

which leads to $n_1 \geq p$ and

$$x_{n+k-1} \in D_{n+k-1} \subset X_2, \qquad \text{for } n \in N(p+1, p+k). \tag{3.12}$$

Thus, it is easy to see $n_1 = p$. Taking $n = p + k$ in (3.9), we have

$$x_{2k+p-1} = g_2^{(k+p)} \cdot g_1^{(k-1)}(x_0) = a^{2k+p-1}x_0 - a^{2k+p-1} + a^{k+p}. \tag{3.13}$$

Let n_2 be the largest integer, such that $x_{n+2k+p-1} \in X_2$, for $n \in N(0, n_2)$. Then, from (1.1) and (3.13), we obtain

$$\begin{aligned} x_{n+2k+p-1} &= g_1^{(n)}(x_{2k+p-1}) = (x_{2k+p-1} - 1)a^n + 1 \\ &= a^{n+2k+p-1}x_0 - a^{n+2k+p-1} + a^{n+k+p} - a^n + 1, \end{aligned} \tag{3.14}$$

for $n \in N(1, n_2 + k)$. This implies that $x_{n+2k+p-1} \in D_{n+2k+p-1}$, for $n \in N(1, n_2 + k)$, where

$$D_{n+2k+p-1} = g_1^{(n)}\left(D_{2k+p-1}\right) = g_1^{(n)} \cdot g_2^{(k+p)} \cdot g_1^{(k-1)}\left(D_0\right). \tag{3.15}$$

Substituting (3.10) with $n_1 = p$ into (3.15), we calculate

$$\begin{aligned} D_{n+2k+p-1} &= \left(g_1^{(n)} \cdot g_2^{(k+p)} \cdot g_1^{(k-1)}(c), g_1^{(n)} \cdot g_2^{(k+p)} \cdot g_1^{(k-1)}(1)\right) \\ &= \left(a^{n+2k+p-1}c - a^{n+2k+p-1} + a^{n+k+p} - a^n + 1, a^{n+k+p} - a^n + 1\right), \end{aligned} \tag{3.16}$$

for $n \in N(1, n_2 + k)$. Since $b \in I_2(p, q)$, it follows from (3.16) that

$$x_{n+2k+p-1} \in D_{n+2k+p-1} \subset X_2, \qquad \text{for } n \in N(0, q), \tag{3.17}$$

and

$$x_{n+2k+p-1} \in D_{n+2k+p-1} \subset D_0 \subset X_3, \qquad \text{for } n \in N\left(q+1, q+k\right), \tag{3.18}$$

which implies that $n_2 = q$.

Taking $n = q + 1$, from (3.18), we have

$$x_{2k+p+q} \in D_{2k+p+q} \subset D_0.$$

From the above facts, we can construct the mapping $g(x) : D_0 \to D_0$ as follows,

$$\begin{aligned} g(x) &= g_1^{(q+1)} \cdot g_2^{(k+p)} \cdot g_1^{(k-1)}(x) \\ &= a^{2k+p+q}x - a^{2k+p+q} + a^{k+p+q+1} - a^{q+1} + 1. \end{aligned} \tag{3.19}$$

Obviously,

$$\lim_{n\to\infty} g^{(n)}(x) = 1 - a^{q+1}\left(1 - a^{k+p}\right)\left(1 - a^{2k+p+k}\right)^{-1} = x^*. \tag{3.20}$$

Because of $c \in I_1(p, q) \cap I_2(p, q)$, we can verify that $x^* \in D_0$. Hence, x^* is the unique fixed point of $g(x)$ in D_0. Clearly, the unique solution $\{x'_n\}$ with initial value $\varphi \in X_2$ and the first iteration $x'_0 = x^*$ is a periodic solution, whose minimal period is $2k + p + q$. From Lemma 3.1 and limit (3.20), we see that the solution $\{x'_n\}$ is a asymptotically stable periodic solution with initial value $\varphi \in X_2 \cup X_3$. This completes the proof of Theorem 3.1.

4. CONCLUSION

In this paper, we have considered the convergence of solutions with the initial function φ of (1.1) for the cases that $b \geq 1$ and $\varphi \in X_1 \cup X_2$; $b \geq 1$ and $c \geq ba^{-k}$ and $\varphi \in X_3$; $0 \leq b < 1 \leq c$ and $\varphi \in X_1$; $0 \leq b < 1 \leq c$ and $\varphi \in X_2$; $0 \leq b < 1 \leq c$ and $\varphi \in X_3$ and $c > ba^{-k}$. We have also proved the existence of asymptotically stable periodic solutions with initial function $\varphi \in X_2 \cup X_3$ and the minimal period $2k + p + q$ of (1.1), for the case that

$$0 \leq b < ba^{-k} < c < 1 \quad \text{and} \quad c \in I_1(p, q) \cap I_2(p, q), \qquad \text{for some } p, q \in N.$$

Under the assumption $\varphi \in X_1 \cup X_2 \cup X_3 = X_{b,c}$, for other cases that were not considered in this paper, one can study the convergence and the periodicity of solutions with the initial function φ for such a class of delay difference equations by using similar techniques as in the present paper, the proofs are quite elementary, but very tedious. If $\varphi \in X$ but $\varphi \notin X_1 \cup X_2 \cup X_3 = X_{b,c}$, that is, $\varphi \in X$ and $\varphi(i) - b$ or $\varphi(i) - c$ can change its sign for $i \in N(-k, -1)$, we expect that the solutions with the initial function φ to this class of equations could exhibit more complicated and interesting large-time behavior, we leave this for future considerations.

REFERENCES

1. K.L. Cooke and J. Wiener, A survey of differential equation with piecewise continuous argument, In *Lecture Notes in Mathematics, Volume 1475*, pp. 1–15, Springer-Verlag, Berlin, (1991).
2. K.L. Cooke and J. Wiener, Retarded differential equations with piecewise constant delays, *J. Math. Anal. Appl.* **99**, 265–297, (1984).
3. S.M. Shah and J. Wiener, Advanced differential equations with piecewise constant argument deviations, *Inter. J. Math. Sci.* **6**, 671–703, (1983).
4. A.R. Aftabizadeh, J. Wiener and J.M. Xu, Oscillatory and periodic solutions of delay differential equations with piecewise constant argument, *Proc. Amer. Math. Soc.* **99**, 673–679, (1987).
5. J. Wiener and S.M. Shah, Functional-differential equation with piecewise constant argument, *Indian J. Math.* **29**, 131–158, (1987).
6. A.R. Aftabizadeh and J. Wiener, Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise constant argument, *Appl. Anal.* **26**, 327–338, (1988).
7. S. Elaydi and S. Zhang, Stability and periodicity of difference equations with finite delay, *Funkcial. Ekvac.* **37**, 401–403, (1994).
8. I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon, Oxford, (1991).
9. S.A. Kurukis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, *J. Math. Anal. Appl.* **188**, 719–731, (1994).
10. V.L. Kocic and G. Ladas, Global attractivity in nonlinear delay difference equations, *Proc. Amer. Math. Soc.* **115**, 1083–1088, (1992).
11. S.A. Levin and R.M. May, A note on difference-delay equations, *Theoret. Population Biol.* **9**, 178–187, (1976).
12. J.H. Shen and I.P. Stavroulakis, Oscillatory and nonoscillatory delay equations with piecewise constant argument, *J. Math. Anal. Appl.* **248**, 385–401, (2000).
13. Y. Wang and J. Yan, Necessary and sufficient condition for the global attractivity of the trivial solution of a delay equation with continuous and piecewise constant arguments, *Appl. Math. Lett.* **10** (5), 91–96, (1997).
14. J.S. Yu and S.S. Cheng, A stability criterion for a neutral difference equation with delay, *Appl. Math. Lett.* **7** (6), 75–80, (1994).
15. J.S. Yu, B.G. Chang, and Z.C. Wang, Oscillation of delay difference equations, *Appl. Anal.* **53**, 117–124, (1994).